Math 33A Worksheet Week 6

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Exercise 1. Let $A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\mathrm{Im} A$. Notice that $\dim \ker A + \dim \mathrm{Im} A = 4$.

Exercise 2. True or false: Explain your reasoning or find an example or counterexample.

(a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3, then $V = \{\vec{0}\}$.

(b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.

(c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.

(d) It is possible for a 4×4 matrix A to have $\ker A = \mathrm{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and

$\mathrm{Im} A = \mathrm{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\rangle$

(e) There exists a 4×4 matrix A with $\ker A = \mathrm{span}\langle e_1, e_2, e_3 \rangle$ and $\mathrm{Im} A = \mathrm{span}\langle e_3 + e_4 \rangle$

(f) There exists a 5×5 matrix A with $\ker A = \mathrm{Im} A$.

(g) There exists a 4×4 matrix A with $\ker A = \mathrm{Im} A$.

Exercise 3. Let $A : \mathbb{R}^8 \to \mathbb{R}^7$ be given by the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $\dim \ker A$ (hint: use rank-nullity and find $\dim \mathrm{Im} A$).

Exercise 4. Find a basis for the following subspaces of $\mathbb{R}^3$:

(a) $V = \mathrm{span}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right)$

(b) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;\middle|\; v_1 - 3v_2 = 0 \right\}$

(c) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be any linear transformation such that $\dim \ker A = 0$. Find a basis for $V = \mathrm{Im} A$.

Exercise 5. For what values of $\lambda \in \mathbb{R}$ are the following pairs of vectors orthogonal?

(a) $\begin{bmatrix} \lambda \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} \lambda \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -\lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

Exercise 6. Consider two subspaces V and W of $\mathbb{R}^n$, where V is contained in W, denoted $V \subseteq W$.

(a) Show that $\dim(V) \leq \dim(W)$.

(b) Show that if $\dim(V) = \dim(W)$, then $V = W$.
